Chaos vs. linear instability in the Vlasov equation: a fractal analysis characterization

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Abstract

In this work we discuss the most recent results concerning the Vlasov dynamics inside the spinodal region. The chaotic behaviour which follows an initial regular evolution is characterized through the calculation of the fractal dimension of the distribution of the final modes excited. The ambiguous role of the largest Lyapunov exponent for unstable systems is also critically reviewed.

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The character of the mean-field dynamics inside the spinodal region has been recently deeply investigated in order to understand the dynamical mechanisms leading to nuclear multifragmentation [1–5]. In this respect the realization of the occurrence of chaotic behaviour in the Vlasov equation [3,4] has been suggested as a possible explanation of the success of statistical models [6,7] in explaining the experimental data. In fact deterministic chaos seems the most natural dynamical mechanism to fill the phase space, a general assumption made by all statistical scenarios. In this work, along the same line of refs. [3,4] we want to characterize in a more precise and clear way the chaotic character of the mean field evolution in the spinodal region which follows an initial linear behaviour.

A generic dynamical system, which evolves according to non-linear equations, is non-integrable and therefore displays chaotic motion if studied for long enough time. The main signature of chaoticity is the extreme sensitivity to the initial conditions, which can be revealed by considering a set of trajectories, initially close in phase space, and plotting the final values of a given dynamical variable vs. the initial ones. If the dynamical system is chaotic, this plot shows an irregular scattered behaviour. A typical example is the plot of the deflection function for chaotic scattering [8,9]. In ref. [4] this procedure is illustrated for two coupled harmonic oscillators.

For the mean field dynamics one can consider a similar procedure. A set of initially close trajectories inside the spinodal region can be calculated and the amplitude of each mode followed in time. If chaos occurs, for a given elapsed time, the final amplitude, plotted against the initial one, should display strong and irregular fluctuations. In the following, after a critical review of the Lyapunov exponents previously calculated [3,4], we discuss scatter plots of the amplitude of the most excited modes during the spinodal evolution [5], which indicates the strength of the coupling between the different modes (i.e. the non-linearity), as well as the sensitivity to the initial conditions. Moreover we quantify the dispersion by calculating the corresponding fractal correlation dimension [10] as a function of time. We show that, within the values of the parameters used in our calculation, the spinodal evolution exhibits an irregular pattern for the lowest modes after 40-50 fm/c, when

the amplitude of the density fluctuations is still a small fraction of the total average density. This behaviour becomes common to all modes after 70-80 fm/c. A full chaotic evolution is reached for all modes after 100 fm/c, when primary "fragments" start to be apparent, thus validating the assumptions of statistical models.

As done in refs. [3,4] the Vlasov equation has been solved numerically in a two-dimensional lattice using the same code of ref. [2], but neglecting the collision integral.

We have studied a fermion gas situated on a large torus with periodic boundary conditions, and its size is kept constant during the evolution. The torus sidelengths are equal to $L_x = 51\ fm$ and $L_y = 15\ fm$. We divide the single particle phase space into several small cells. We employed in momentum space 51x51 small cells of size $\Delta p_x = \Delta p_y = 40\ MeV/c$, while in coordinate space $\Delta x = 0.3333\ fm$ and $\Delta y = 15\ fm$, *i.e.* we have only one big cell on the y-direction. The initial local momentum distribution was assumed to be the one of a Fermi gas at a fixed temperature $T = 3\ MeV$. We employ a local Skyrme interaction which generates a mean field $U[\rho] = t_0\ (\rho/\rho_0) + t_3\ (\rho/\rho_0)^2$. The density ρ is folded along the x-direction with a gaussian of width $\mu = 0.61 fm$, in order to give a finite range to the interaction. The parameters of the force t_0 and t_3 have been chosen in order to reproduce correctly the binding energy of nuclear matter at zero temperature, and this gives $t_0 = \ -100.3\ MeV$ and $t_3 = \ 48\ MeV$. The resulting EOS gives a saturation density in two dimensions equal to $\rho_0 = \ 0.55\ fm^{-2}$ which corresponds to the usual three-dimensional Fermi momentum equal to $P_F = \ 260\ MeV/c$. Then a complete dynamical evolution is performed by subdividing the total time in small time steps, each equal to $\Delta t = 0.5\ fm/c$.

For more details concerning the mean field propagation on the lattice, the reader is referred to ref. [2].

In refs. [3,4] the density of the system was initially perturbed mainly by means of small sinusoidal waves with random noise in order to study the response to minor change in the initial conditions. It was shown that a very sensitive dependence on the initial conditions appears as soon as the average density is inside the spinodal region. We also found positive Lyapunov exponents, confirming that the dynamical evolution is chaotic. In this respect an

important clarification must be done. In fact it turns out that if the initial density is an exact harmonic oscillation, then the evolution is regular for very long times [11]. This seems to be a very general statement valid also for molecular dynamics approaches as far as only eigenmodes are initially excited [12]. However it should be stressed that one can initialize the density according to several different prescriptions. In this respect the exact eigenmode is quite a special case, whereas a perturbed sinusoidal wave is a possible and even more realistic way of starting the system in order to study its response.

A further clarification concerns the role of the Lyapunov exponents extracted in our previous works. It turns out that the values obtained are very similar to the growth factor of the fastest mode obtained in the analytical linear response [5,11].

The above considerations could be misleading and make the reader think that the conclusions of refs. [3,4] are erroneous. Actually this is not the case and we want to discuss it further in the following.

Let us consider a bunch of several randomly initialized trajectories and the distance

$$d(t) = \big|\ |F_k(t)|_1 - |F_k(t)|_2\ \big| \quad , \tag{1}$$

being $|F_k(t)|_1$ and $|F_k(t)|_2$ the modulus of the strength of the k mode excited for the trajectory 1 and 2. In fig.1 we plot the evolution of $d(t)/d_0$, where $d_0 = d(t=0)$, for 9 different trajectories. The modes $n_k = 3, 4, 5$ and $n_k = 12, 13, 14$ are considered at an initial density $\rho/\rho_0 = 0.5$. We should remind that the linear approximation to the Vlasov equation produces a dispersion relation $\omega = \omega(n_k)$ for all modes below the cut-off, which depends on the used interaction range. In our simulation the modes $n_k = 12, 13, 14$ are the fastest ones, whereas the modes $n_k = 3, 4, 5$ evolve more slowly. In fig.1 it is evident a linear and predictable growth up to a time which vary between 30-40 fm/c and 70-80 fm/c respectively for the smallest and the largest modes excited in the simulations. Figure 1 clearly shows that the time scale of the linear evolution is limited to the first part. The latter depends on the modes. This was already discussed in ref. [4] (see fig.7) but for one simulation only.

It should be noted that this metric (eq.1) takes into account the single contributions of

the k modes to the complete evolution, whereas according to the metric adopted previously in ref. [3,4] the contribution of all modes was taken on average.

From this simulation one can also extract the Lyapunov exponent. In figure 1 it appears that, especially for the low k modes, after the linear regime, the distance between trajectories has a quite irregular trend as a function of time. The average slope appears to be much larger than the one of the linear regime, and in some cases it can display an "up and down" behaviour. All that reflects the strong coupling among all the modes. It is therefore meaningful to analyze separately the two regimes and to extract the average slopes for each one of the two. In the corresponding interval this gives an average value of the Lyapunov exponent λ, being $d = d_0 e^{\lambda t}$. The values extracted are reported in table 1, where the interval of time in which the average has been taken is explicitely indicated. In this case the values extracted mode by mode are by far different from the growth linear factor - also reported - and only taking the average between all the modes excited one gets a value which is similar to the linear growth of the largest exponent.

This result implies that the precise value of the Lyapunov exponent is not able to discriminate clearly between a chaotic and a regular behaviour in the present case of an unstable system. However taking into account this result together with the broad power spectra of the excited modes, found in ref. [3,4], and the sensitive dependence on the initial condition leaves no room for a linear behaviour once the density fluctuations reach an appreciable amplitude. It should be noticed that chaotic behaviour is a generic statement adopted in the literature. In general it does not mean a full randomness, which is the extreme limit, but only an irregular motion. The Lyapunov exponent is one way to quantify this irregularity and unpredictability.

A further way to quantify how chaotic is the system is by considering final observable quantities as a function of initial one, as discussed in the introduction. On the right-hand side of figure 2 we plot in the case of 500 random-initialized simulations and for $\rho/\rho_0 = 0.5$ the final modulus of the strength $|F_k(t)|_f$ as a function of the initial one $|F_k(t)|_i$. To help understanding the evolution we plot aside the complete numerical evolution for one event.

The change in the evolution from a linear character to a chaotic one emerges clearly from the figure. This variation is faster for the smaller modes than for the biggest, as previously discussed. In a recent preprint [5] similar scatter plots are presented. Our calculations were performed independently of those results.

In order to quantify this irregular dynamics we have calculated by means of the Grassberger-Procaccia algorithm [10] the fractal correlation dimension D_2 [8]. That is we calculate the correlation integral

$$C(r) = \frac{1}{M^2} \sum_{i,j}^{M} \Theta(r - |\mathbf{z}_i - \mathbf{z}_j|) \tag{2}$$

being Θ the Heaviside step function and $\mathbf{z}_i$ a vector whose two components (x_i, y_i) are the initial and final strengths of the modes k plotted in fig.2. When r is small one gets

$$C(r) \cong r^{D_2} \qquad\qquad a << r << L, \tag{3}$$

being a and L the minimum and the maximum size of the set of points considered. Thefore by plotting the logarithms it is possible to extract D_2. As an example we plot in fig.3 $log\ C(r)$ vs. $log\ r$ for the mode $n_k = 5$ at t=120 fm/c. We considered as a good interval for the scaling behaviour satisfying eq.(3) the one between $r_{min} = 7.5 \cdot 10^{-5}$ and $r_{max} = 1.5 \cdot 10^{-3}$. The dashed curve is the fit whose slope gives $D_2 = 1.97 \pm 0.05$. The error one gets is of statistical nature and is related to the limited number of points considered. In order to improve the accuracy of the numerical algorithm the y_i coordinate was normalized to the x_i. This was necessary being the two scales quite different.

The general behaviour of D_2 as a function of time is shown in fig.4 for the modes $n_k = 5$ and $n_k = 14$. The dashed lines are to guide the eye. The figure shows, in line with the previous discussion, that after an initial linear dynamics which depends on the mode and in any case is not longer than 70-80 fm/c the evolution becomes more and more chaotic till a complete randomness. In fact $D_2 = 2$ is what one would expect and actually gets - we have tested our algorithm in this sense - when distributing random numbers on a plane.

In conclusion, the present investigation confirms without any doubt that after some short time which of course depends on the parameters used - for example the range of the force

adopted - a fully chaotic behaviour occurs in the Vlasov dynamics inside the spinodal region. This result confirms the crucial role played by deterministic chaos in filling the phase space, a fundamental assumption for justifying the validity of statistical approaches. On the other hand the success of the latter in explaining most of the experimental data is a confirmation of this scenario. However when the fragmentation regime is very fast and chaos is not fully developed, some memory of the modes initially excited can probably remain [5]. We note in passing that a chaotic behaviour with respect to multifragmentation has been recently found also in other models [13,14].

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REFERENCES

[1] H. Heiselberg C.J. Pethick and D.G. Ravenhall, *Phys. Rev. Lett.* **61**, 818 (1988) and *Ann. Phys.* **223**, 37 (1993).

[2] G.F. Burgio, Ph. Chomaz and J. Randrup, *Phys. Rev. Lett.* **69**, 885 (1992).

[3] G.F. Burgio, M. Baldo and A. Rapisarda, *Phys. Lett.* **B321**, 307 (1994); M. Baldo, G.F. Burgio and A. Rapisarda, *Nucl. Phys.* **A583**, 343 (1995).

[4] M. Baldo, G.F. Burgio and A. Rapisarda, *Phys. Rev.* **C51**, 198 (1995).

[5] B. Jacquot, A. Guarnera, Ph. Chomaz and M. Colonna, GANIL preprint P95-17, *Phys. Rev.* **C** submitted

[6] D.H.E. Gross, *Nucl. Phys.* **A553**, 175c (1993) and references therein.

[7] J.P. Bondorf, A.S. Botvina, A.S. Iljinov, I.N. Mishustin and K. Sneppen, *Phys. Rep.* **257**,133 (1995).

[8] E. Ott, *Chaos in Dynamical Systems*, Cambridge University Press, Cambridge, 1993.

[9] A. Rapisarda and M. Baldo, *Phys. Rev. Lett.* **66**, 2581 (1991); M. Baldo, E.G. Lanza and A. Rapisarda, *Chaos* **3**, 691 (1993).

[10] P. Grassberger and I.Procaccia, *Phys. Rev. Lett.* **50**, 346 (1983).

[11] B. Jacquot, M. Colonna, Ph. Chomaz and A. Guarnera, GANIL preprint P95-15, *Phys. Lett.* **B**, in press.

[12] J.J. Binney, N.J. Dowrick, A.J. Fisher and M.E.J. Newman, *The Theory of Critical Phenomena*, Oxford Science Publications, 1993.

[13] H.Feldmeier, GSI report 02-1994.

[14] A. Bonasera, V. Latora and A. Rapisarda, preprint LNS-5-5-95, *Phys. Rev. Lett.* submitted.

TABLES

n_k	t_l (fm/c)	t_r (fm/c)	$\lambda * 10^2$ (c/fm)	$\omega_k * 10^2$ (c/fm)
3	25.	50.	2.3	2.3
3	50.	105.	7.0	
4	25.	50.	2.6	3.0
4	50.	105.	9.3	
5	25.	50.	3.4	3.7
5	50.	105.	8.4	
12	50.	90.	6.0	6.9
12	90.	105.	4.7	
13	50.	90.	6.1	7.1
13	90.	105.	3.9	
14	50.	90.	6.0	7.0
14	90.	105.	4.7	
≤ 24	50.	105.	5.6	6.3

TABLE I. For several n_k modes, we show a comparison between the Lyapunov exponent λ and the frequency ω predicted by the linear response approximation. As indicated, λ depends on the time interval $\Delta t = t_r - t_l$ over which the average slope is estimated. t_l and t_r are reported in the second and third column. Please note the last row, where λ and ω are compared in the case the metric contains the contributions of all modes summed up to $n_k =$ 24.

FIGURES

FIG. 1. The time evolution of $d(t)/d_0$ is shown for different modes and several trajectories at an initial density $\rho/\rho_0 = 0.5$. On the left-hand side we display the behaviour of the slowest n_k's, whereas on the right-hand side the behaviour of the fastest modes is plotted.

FIG. 2. On the left-hand side we show the time evolution of a density profile vs. the position. The initial average density is half normal nuclear matter density. On the right-hand side we display scatter plots of the Fourier spectra of a bunch of 500 trajectories. Only two modes $n_k = 5$ and $n_k = 14$ are shown. See text for more details.

FIG. 3. The log of the correlation integral C(r) is shown for $n_k = 5$ at time $t = 120\ fm/c$ as a function of the log of the distance parameter r according to the definition of eq.(2). The circles are the results of the numerical simulations and the dashed line is a linear fit.

FIG. 4. The time evolution of the correlation dimension D_2 is plotted for the modes $n_k = 5$ and $n_k = 14$.